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A CHARACTERIZATION OF SUPERCOMPACTNESS WITH AN APPLICATION TO TREELIKE SPACES

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A characterization of supercompactness with an application to treelike spaces

by

A.E. Brouwer & A. Schrijver

ABSTRACT

The concept of interval structure is introduced and a characterization of supercompactness is given in terms of interval structures. This characterization is used to prove the supercompactness of a compact treelike space.

KEY WORDS AND PHRASES: *interval structure, supercompact, treelike.*

1. SUPERCOMPACTNESS

In this section we give definitions of supercompact spaces and interval structures and a criterion for supercompactness with help of interval structures.

DEFINITION. Let X be a set and $\mathcal{S}$ a subset of the powerset $\mathcal{P}(X)$. Then $\mathcal{S}$ is called *binary* if for each nonempty $\mathcal{S}' \subset \mathcal{S}$ with $\cap\mathcal{S}' = \emptyset$ there exist S_1 and S_2 in $\mathcal{S}$ such that $S_1 \cap S_2 = \emptyset$.

DEFINITION. A topological space X is called *supercompact* if there exists a binary closed subbase for X.

By ALEXANDER's lemma it can be easily seen that every supercompact space is compact.

DEFINITION. Let X be a set and $I: X \times X \to \mathcal{P}(X)$. Write $I(x,y) = I((x,y))$. Then I is called an *interval structure* on X if:

(i) $x,y \in I(x,y)$, $(x,y \in X)$,

(ii) $I(x,y) = I(y,x)$, $(x,y \in X)$,

(iii) if $u,v \in I(x,y)$ then $I(u,v) \subset I(x,y)$, $(u,v,x,y \in X)$,

(iv) $I(x,y) \cap I(x,z) \cap I(y,z) \neq \emptyset$, $(x,y,z \in X)$.

Axioms (i), (ii) and (iii) together can be replaced by the following axiom:

$$u,v \in I(x,y) \text{ iff } I(u,v) \subset I(y,x) \qquad (x,y,u,v \in X).$$

Examples of interval-structures:

a. if $(X,\leq)$ is a lattice, then $I(x,y) = \{z \mid x \wedge y \leq z \leq x \vee y\}$ defines an interval-structure;

b. if X is a treelike space, then $I(x,y) = \{z \mid z \text{ separates } x \text{ and } y\} \cup \{x,y\}$ defines an interval-structure (see section 3).

DEFINITION. Let I be an interval structure on the set X and $X' \subset X$. X' is *I-closed* if for each $x,y \in X'$ $I(x,y) \subset X'$.

THEOREM 1.1. *Let* X *be a topological space. Then:*
X *is supercompact if and only if* X *is compact and there exists a closed subbase* $\mathcal{S}$ *and an interval structure* I *such that every* $S \in \mathcal{S}$ *is* I*-closed.*

PROOF.

(a) Let X be a supercompact space and let $\mathcal{S}$ be a binary closed subbase for X. Define $I: X \times X \to \mathcal{P}(X)$ by

$$I(x,y) = \bigcap\{S \in \mathcal{S} \mid x,y \in S\}, \qquad (x,y \in X).$$

Then I is an interval structure on X and each $S \in \mathcal{S}$ is clearly I-closed. To prove the former, we will only show that for each $x,y,z \in X$ $I(x,y) \cap I(x,z) \cap I(y,z) \neq \emptyset$. By the definition of I: $I(x,y) \cap I(x,z) \cap I(y,z) = \bigcap\{S \in \mathcal{S} \mid \{x,y,z\} \cap S$ contains two or more elements$\}$. Suppose this intersection is empty. Then, since $\mathcal{S}$ is binary, there exist S_1 and S_2 in $\mathcal{S}$ such that $\{x,y,z\} \cap S_1$ and $\{x,y,z\} \cap S_2$ both contain two or more elements and $S_1 \cap S_2 = \emptyset$, which is a contradiction.

(b) Conversely, let X be a compact space with closed subbase $\mathcal{S}$, and let I be an interval structure on X, such that each $S \in \mathcal{S}$ is I-closed. We prove that $\mathcal{S}$ is binary.
Let $\mathcal{S}' \subset \mathcal{S}$ be such that $\bigcap\mathcal{S}' = \emptyset$. Then, since X is compact, there exists a finite subset $\mathcal{S}'_0 \subset \mathcal{S}'$ such that $\bigcap\mathcal{S}'_0 = \emptyset$. Hence it is enough to prove the following: if $S_1,\dots,S_k \in \mathcal{S}$ and $S_1 \cap \dots \cap S_k = \emptyset$ then there exist i,j $(1 \le i,j \le k)$ such that $S_i \cap S_j = \emptyset$.
We proceed by induction with respect to k. If k = 1 or 2 it is trivial. Suppose $k \ge 3$ and for each $k' < k$ the statement is true.
Define:

$$\begin{aligned} T_1 &= \phantom{S_1 \cap{}} S_2 \cap S_3 \cap S_4 \cap \dots \cap S_k, \\ T_2 &= S_1 \cap \phantom{S_2 \cap{}} S_3 \cap S_4 \cap \dots \cap S_k, \\ T_3 &= S_1 \cap S_2 \cap \phantom{S_3 \cap{}} S_4 \cap \dots \cap S_k. \end{aligned}$$

If one of these is empty, then the induction hypothesis applies.
Suppose therefore $T_i \neq \emptyset$ $(i=1,2,3)$, and take $x \in T_1$, $y \in T_2$ and $z \in T_3$.
Then

$$\begin{aligned} x,y &\in S_3 \cap S_4 \cap \dots \cap S_k, \\ x,z &\in S_2 \cap S_4 \cap \dots \cap S_k, \\ y,z &\in S_1 \cap S_4 \cap \dots \cap S_k, \end{aligned}$$

and thus

$$\begin{aligned} I(x,y) &\subset S_3 \cap S_4 \cap \dots \cap S_k, \\ I(x,z) &\subset S_2 \cap S_4 \cap \dots \cap S_k, \end{aligned}$$

$$I(y,z) \subset S_1 \cap S_4 \cap \ldots \cap S_k,$$

But $I(x,y) \cap I(x,z) \cap I(y,z) \neq \emptyset$, so that

$$(S_1 \cap S_4 \cap \ldots \cap S_k) \cap (S_2 \cap S_4 \cap \ldots \cap S_k) \cap (S_3 \cap S_4 \cap \ldots \cap S_k) =$$
$$= S_1 \cap S_2 \cap S_3 \cap S_4 \cap \ldots \cap S_k \neq \emptyset.$$

This contradicts our hypothesis. □

For some related ideas see GILMORE [1].

2. TREELIKENESS

In this section we recall the definition of treelike spaces and mention some of their properties.

DEFINITION. A topological space X is called *treelike* if it is connected and for any two points x,y there is a point z separating x and y. Notation: $X \backslash z = \underset{x}{A} + \underset{y}{B}$ means that $X\backslash\{z\}$ can be written as the topological sum of two subspaces A and B, containing x and y respectively.

PROPOSITION 2.1. *A treelike space is Hausdorff.*
If X is treelike and $x,y \in X$ we set $E(x,y) := \{z \mid z \text{ separates } x \text{ and } y \text{ in } X\}$ and $S(x,y) := E(x,y) \cup \{x,y\}$.

PROPOSITION 2.2. *Let* X *be treelike and* $x,y \in X$. *Then* $S(x,y)$ *can be ordered in a natural way by setting* $x \leq y$ *and* $p < y$ *for* $p \in E(x,y)$ *and* $p < q$ *if* q *separates* p *and* y *for* $p,q \in E(x,y)$. *This order contains no jumps and no gaps.*

PROPOSITION 2.3. *If* X *is a treelike space and* $p \in X$ *then all components of* $X\backslash p$ *are open in* X.

PROPOSITION 2.4. *If* X *is treelike and either locally connected (cf.* WHYBURN [5]) *or locally peripherally compact (cf.* PROIZVOLOV [4]) *then for all* $x,y \in X$ $S(x,y)$ *is connected.*

The above results are well-known and can be found scattered through the literature in various forms. In many older papers separable metrizability is required. It seems that the paper of WHYBURN [5] was the first one explicitly dropping this condition. In KOK [2], a coherent account is given of the implications and interrelations of many properties of spaces, among them being treelikeness (which he calls property (S)). The following lemma from [1] will be needed in the next section.

LEMMA 2.5. *Let* X *be a connected topological space,* $C \subset X$ *connected,* S *a component of* $X \setminus C$. *Then* $X \setminus S$ *is connected.*

3. A COMPACT TREELIKE SPACE IS SUPERCOMPACT

In this section we first show that on each treelike space an interval structure can be defined, and next that a compact treelike space is supercompact.

PROPOSITION 3.1. *Let* X *be a treelike space. Then* $I(x,y) = S(x,y)$ *defines an interval structure on* X.

PROOF.

(i) $x,y \in S(x,y)$ by definition.

(ii) $S(x,y) = S(y,x)$ by definition.

(iii) If z separates x and y: $X \setminus z = \underset{x}{A} + \underset{y}{B}$ then $\bar{A} = A \cup \{z\}$ and $\bar{B} = B \cup \{z\}$ are both connected. Therefore if u separates x and z then $u \in A$ and B is contained in one component of $X \setminus u$, i.e. u separates x and y. This proves $z \in S(x,y) \Rightarrow S(x,z) \subset S(x,y)$.

(iv) Suppose $S(x,y) \cap S(y,z) \cap S(x,z) = \emptyset$. By definition $S(x,y) \subset S(y,z) \cup$ $\cup\, S(x,z)$. Let $E := S(x,y) \cap S(y,z)$ and $F := S(x,y) \cap S(x,z)$. E and F are intervals in the order of $S(x,y)$ and $e > f$ for all $e \in E$, $f \in F$. Since $S(x,y)$ contains no gaps, either E contains a first, or F contains a last element. Suppose u is the first element of E. Now $S(y,z) \cup \{u\} =$ $= S(y,u) \cup S(u,z)$. (Because $v \in E(y,u) \Rightarrow v \in E(x,y) \setminus E(x,z) \Rightarrow v \in E(y,z)$ and conversely $v \in E(y,z) \setminus E(y,u) \Rightarrow v \in E(u,z)$.) But this would imply that $S(y,z) = S(y,u) \setminus u + S(u,z) \setminus u$ contained a gap. Contradiction. (Cf. H. KOK [2] pp.45-50). $\square$

NOTE: We need this proposition only in the case that X is compact, in which

case a much shorter proof of (iv) can be given, namely:
Suppose $S(x,y) \cap S(y,z) \cap S(x,z) = \emptyset$. Since $S(p,q)$ is closed and connected for $p,q \in X$ we have $S(x,y) = S(x,y) \underset{y}{\cap} S(y,z) + S(x,y) \underset{x}{\cap} S(x,z)$, a contradiction.

THEOREM 3.2. *Let* X *be a compact treelike space. Then* X *is supercompact.*

PROOF. Using theorem 1.1 and proposition 3.1 it is sufficient to exhibit a closed subbase $\mathcal{S}$ consisting of connected sets.
Claim: $\mathcal{S} := \{X\backslash C \mid p \in X, C \text{ component of } X\backslash p\}$ is such a subbase.
First, by proposition 2.3 each $S \in \mathcal{S}$ is closed. Next, by lemma 2.5 each $S \in \mathcal{S}$ is connected. If $x,y \in X$ and p separates x and y : $X\backslash p = A + B +$ + other components, where A and B are connected, then A and B are disjoint neighbourhoods of x and y in the topology $\mathcal{T}$ generated by $\mathcal{S}$, which is therefore Hausdorff. Since this topology is weaker than the original compact topology on X, both topologies coincide. □

This last result has been proved independently (using a different method) by J. VAN MILL [3].

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